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Optimality Conditions and Adjoint State for a Perturbed Boundary Optimal Control System

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Abstract—In this paper, we are concerned with finding optimal controls for a class of linear boundary optimal control systems associated to a Laplace operator on a regular bounded domain in the n-dimensional Euclidean space. For these systems, in previous works (see [1,2]), we proved existence of the (perturbed) states and optimal controls, and studied their behaviour. The purpose of this paper is to establish the system of optimality conditions, investigate the adjoint states, and prove their strong convergence in some Sobolev spaces.

Keywords—Perturbed states, Linear boundary optimal control systems, Optimality conditions, Adjoint state.

1. NOTATIONS AND STATEMENT OF THE PROBLEMS

1.1. Notations

Let Ω be a connected and simply connected regular, and bounded open subset of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega := \Gamma$. We denote $H^1(\Omega)$ the real classical Sobolev space equipped with its usual inner product and associated norm $\|.\|_{H^1(\Omega)}$. We recall (see, for example, [3]), that the inner product of $H^1(\Omega)$ given by

$$(y \mid z) = \int_\Omega \nabla y . \nabla z \, d\omega + \int_\Gamma yz \, d\gamma, \qquad y, z \in H^1(\Omega), \tag{1}$$

is equivalent to the usual inner product of $H^1(\Omega)$. For every $y \in H^1(\Omega)$, we will denote by $T(y)$ or simply by y the $\Gamma-$trace (i.e., the restriction of y to Γ). We know, by the trace theorem, see [4], that the map T is a bounded linear map from the Sobolev space $H^1(\Omega)$ to $L^2(\Gamma)$.

1.2. Recalls and Statement of the Problems

In the paper [1], for every $\epsilon > 0$, we were concerned by finding

$$\min \left\{ J_\epsilon(v); v \in \mathcal{U}_{ad} \right\}, \tag{Q_ϵ}$$

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where $\mathcal{U}_{ad}$ is any arbitrary finite dimensional subspace of $L_0^2(\Gamma) := \{u \in L^2(\Gamma) : \int_\Gamma u\, d\gamma = 0\}$, and

$$J_\epsilon(v) = \int_\Gamma (y_\epsilon(v) - h_1)^2\, d\gamma + \int_\Gamma \left(\frac{\partial}{\partial\nu}\, y_\epsilon(v) - h_2\right)^2 d\gamma,$$

where h_1, h_2 are two fixed (decision) functions in $L^2(\Gamma)$, and $y_\epsilon(v)$ is a solution of the following problem:

$$\begin{aligned} -\Delta y_\epsilon(v) &= 0, && \text{on } \Omega, \\ \frac{\partial}{\partial\nu}\, y_\epsilon(v) + \epsilon y_\epsilon(v) &= v, && \text{at } \Gamma = \partial\Omega, \\ y_\epsilon(v) &\in V, \end{aligned} \qquad (P_\epsilon)(v)$$

where $\frac{\partial}{\partial\nu}\, y_\epsilon(v)$ is the normal derivative of $y_\epsilon(v)$, and V is the space given by

$$V = \left\{ y \in H^1(\Omega) : \int_\Gamma y\, d\gamma = 0 \right\}.$$

The space V is a Hilbert space when it is endowed with the restriction of the Hilbert structure of $H^1(\Omega)$. We can also consider in V the following inner product and its associated norm given by:

$$\langle y \mid z \rangle = \int_\Omega \nabla y . \nabla z\, d\omega, \quad \|y\|_V = \left[\int_\Omega |\nabla y|^2\, d\omega \right]^{1/2}, \qquad y, z \in V. \tag{2}$$

It is well known that the norm $\|.\|_V$ is equivalent to the restriction of the usual norm of $H^1(\Omega)$ to the space V. Therefore, we can find a constant $\lambda > 0$ such that

$$\|y\|_{L^2(\Gamma)} \leq \lambda \|y\|_V, \qquad \forall y \in V. \tag{3}$$

1.3. A Convergence Result

In the case where $\mathcal{U}_{ad}$ is any arbitrary finite dimensional subspace of $L_0^2(\Gamma)$, we have proved in [1] existence and uniqueness of the state y_ϵ and the optimal control u_ϵ for the system $(P_\epsilon)(u_\epsilon)$ and (Q_ϵ). We studied also their behaviour. But our study made in that particular case can be generalized to any arbitrary closed convex subset of $L_0^2(\Gamma)$, and the following result will be proved in the next section.

THEOREM 1.3. *Let $\mathcal{U}_{ad}$ be any arbitrary closed and convex subset of $L_0^2(\Gamma)$, and let $0 < \epsilon < 1$. Then we have the following.*

(1.3.1) *There exists a unique element $u_\epsilon \in L_0^2(\Gamma)$, verifying $J_\epsilon(u_\epsilon) = \min\{J_\epsilon(v); v \in \mathcal{U}_{ad}\}$, where $J_\epsilon(v) = \int_\Gamma (y(v) - h_1)^2\, d\gamma + \int_\Gamma (\frac{\partial}{\partial\nu}\, y(v) - h_2)^2\, d\gamma$, and $y_\epsilon(v)$ is the unique solution of the problem $(P_\epsilon)(v)$.*

(1.3.2) *The net $(u_\epsilon)_\epsilon$ of optimal controls converges strongly in $L^2(\Gamma)$, to the unique element $u \in \mathcal{U}_{ad}$, verifying $J(u) = \min\{J(v); v \in \mathcal{U}_{ad}\}$, where $J(v) = \int_\Gamma (y(v) - h_1)^2\, d\gamma + \int_\Gamma (\frac{\partial}{\partial\nu}\, y(v) - h_2)^2\, d\gamma$, and $y(v)$ is the unique solution of the following problem:*

$$\begin{aligned} -\Delta y(v) &= 0, && \text{on } \Omega, \\ \frac{\partial}{\partial\nu}\, y(v) &= v, && \text{at } \Gamma = \partial\Omega, \\ y(v) &\in V, \end{aligned} \qquad (P_0)(v)$$

(1.3.3) *The net $(y_\epsilon(u_\epsilon))_\epsilon$ of states converges in the space V to $y(u)$, the unique solution of the problem $(P_0)(u)$.*

1.4. Organization of the Paper

After proving Theorem 1.3 in the second section, we investigate the system of optimality conditions for the optimal controls of the problem (Q_ϵ), and find the adjoint state p_ϵ associated to the perturbed state y_ϵ. This is done in Section 3. In Section 4, we establish the strong convergence of the adjoint state p_ϵ to the adjoint state p associated to y the solution of the system $(P_0)(u)$, where u is the optimal control involved in Theorem 1.3.

2. PROOF OF THEOREM 1.3

2.1. Step One

The uniqueness of u_ϵ results from the fact that J_ϵ is strictly convex. The existence of u_ϵ is clear when $\mathcal{U}_{ad}$ is bounded. When $\mathcal{U}_{ad}$ is not bounded in $L_0^2(\Gamma)$, then by a classical result of Lions (see [5]) in order to prove the existence of optimal controls, it suffices to verify the two following conditions.

(i) The map $v \to J_\epsilon(v)$ is weakly l.s.c. (i.e., lower semicontinuous) on the set $\mathcal{U}_{ad}$.
(ii) For every sequence (v_n) in $\mathcal{U}_{ad}$, such that $\|v_n\|_{L^2(\Gamma)} \longrightarrow +\infty$, then $J_\epsilon(v_n) \longrightarrow +\infty$, when $n \longrightarrow +\infty$.

But (ii) is evident, and (i) is a consequence from the fact that the linear mapping $R_\epsilon : L_0^2(\Gamma) \longrightarrow L^2(\Gamma)$, $v \longrightarrow y_\epsilon(v)$ is injective and compact having a norm $\|R_\epsilon\| \le \lambda^2$.

2.2. Step Two

Let $w \in \mathcal{U}_{ad}$ be fixed. Then for every $\epsilon \in]0,1[$, we have $0 \le J_\epsilon(u_\epsilon) \le J_\epsilon(w)$. It is easy to see that $(y_\epsilon(w))_\epsilon$ is bounded in $L^2(\Gamma)$. Therefore, the net $(J_\epsilon(w))_\epsilon$ is bounded in $\mathbb{R}$. Then one can find two positive constants C_1 and C_2 such that $\|y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)} \le C_1$, and $\|\frac{\partial}{\partial\nu} y_\epsilon(u_\epsilon)\|_{L^2(\Gamma)} \le C_2$. Thus, we can say that there exists a positive constant C_3 such that $\|u_\epsilon\|_{L^2(\Gamma)} \le C_3$, independently of all $\epsilon \in]0,1[$. Hence, we can find a subsequence (called again $(u_\epsilon)_\epsilon$) converging weakly to a unique element $u \in \mathcal{U}_{ad}$.

2.3. Step Three

Let us denote u_* the unique element in $\mathcal{U}_{ad}$ verifying $J(u_*) = \min\{J(v) : v \in \mathcal{U}_{ad}\}$, where $J(v) = \int_\Gamma (y(v) - h_1)^2\, d\gamma + \int_\Gamma (\frac{\partial}{\partial\nu} y(v) - h_2)^2\, d\gamma$, and $y(v)$ is the unique solution of the problem $(P_0)(v)$. To simplify the notations, we set $y_\epsilon(u_\epsilon) = y_\epsilon$. It is easy to verify that the following inequality holds true:

$$\|y(v) - y_\epsilon(v)\|_V \le \epsilon\lambda^3 \|v\|_{L^2(\Gamma)}, \qquad \forall\epsilon > 0. \tag{4}$$

From (4) we deduce the following inequality:

$$\limsup_{\epsilon\to 0} J_\epsilon(u_\epsilon) \le J(v), \qquad \forall v \in \mathcal{U}_{ad}. \tag{5}$$

Since y_ϵ is bounded in V, we can find a subsequence (denoted again by y_ϵ) converging weakly to an element $z \in V$. It is not hard to see that we must have $z = y(u)$, where $y(u)$ is the unique solution of the problem $(P_0)(u)$. Now, by using [3, Theorem 4, p. 143], we see that we can suppose that this subsequence converges also strongly to $y(u)$ in the space $L^2(\Gamma)$. Then according to the lower semicontinuity of the norm in $L^2(\Gamma)$, we can assert that

$$\liminf_{\epsilon\to 0} J_\epsilon(u_\epsilon) \ge J(u). \tag{6}$$

Then we deduce that $u = u_*$ and that

$$\lim_{\epsilon\to 0} J_\epsilon(u_\epsilon) = J(u) = \int_\Gamma |y(u) - h_1|^2\, d\gamma + \int_\Gamma |u - h_2|^2\, d\gamma.$$

Now, since the sequence y_ϵ (of traces) converges in $L^2(\Gamma)$ to (the trace) $y(u)$. Therefore, we obtain that

$$\lim_{\epsilon\to 0} \int_\Gamma \left|\frac{\partial}{\partial\nu} y_\epsilon - h_2\right|^2 d\gamma = \int_\Gamma |u - h_2|^2\, d\gamma,$$

from which we obtain the convergence of the sequence $\|u_\epsilon\|_{L^2(\Gamma)}$ to $\|u\|_{L^2(\Gamma)}$ when $\epsilon \longrightarrow 0$, and hence, that the sequence u_ϵ converges strongly in $L^2(\Gamma)$ to u. We can deduce also that the whole net u_ϵ converges strongly in $L^2(\Gamma)$ to u, where u is the unique element $u \in \mathcal{U}_{ad}$, verifying $J(u) = \min\{J(v); v \in \mathcal{U}_{ad}\}$, where $J(v) = \int_\Gamma (y(v) - h_1)^2\, d\gamma + \int_\Gamma (\frac{\partial}{\partial\nu} y(v) - h_2)^2\, d\gamma$, and $y(v)$ is the unique solution of the problem $(P_0)(v)$. This proves (1.3.1) and (1.3.2).

2.4. Step Four

In order to show that the whole net $(y_\epsilon)_\epsilon$ converges in V to $y(u)$, we use the following inequalities:

$$\begin{aligned} \|y_\epsilon(u_\epsilon) - y(u)\|_V &\le \|y_\epsilon(u_\epsilon) - y_\epsilon(u)\|_V + \|y_\epsilon(u) - y(u)\|_V \\ &\le \lambda \|u_\epsilon - u\|_{L^2(\Gamma)} + \|y_\epsilon(u) - y(u)\|_V . \end{aligned} \tag{7}$$

Since $(u_\epsilon)_\epsilon$ converges strongly to u in $L^2(\Gamma)$ and according to (4), we see in (7) that the state $(y_\epsilon(u))$ converges to $y(u)$ in V when $\epsilon \longrightarrow 0$. This finishes the proof of our theorem. ∎

3. ADJOINT STATE AND SYSTEM OF OPTIMALITY CONDITIONS

3.1. Adjoint State

In the sequel of this paper, we shall suppose that $\mathcal{U}_{ad} = L_0^2(\Gamma)$. Let $0 < \epsilon < 1$. For every $v \in L_0^2(\Gamma)$, we consider the following system:

$$-\Delta p_\epsilon(v) = 0, \qquad \text{on } \Omega,$$

$$\frac{\partial}{\partial \nu} p_\epsilon(v) + \epsilon p_\epsilon(v) = y_\epsilon(v) - h_1 + \frac{1}{|\Gamma|}\int_\Gamma h_1\, d\gamma - \epsilon\left(\frac{\partial}{\partial \nu} y_\epsilon(v) - h_2\right) - \frac{\epsilon}{|\Gamma|}\int_\Gamma h_2\, d\gamma, \qquad (P_\epsilon)^*(v)$$

$$\text{at } \Gamma = \partial\Omega, \qquad p_\epsilon(v) \in V,$$

where $|\Gamma|$ designates the Lebesgue measure of Γ, and $y_\epsilon(v)$ is the solution of the system $(P_\epsilon)(v)$. As before, one can use Lax-Milgram Theorem in the variational formulation for the problem $(P_\epsilon)^*(v)$, and conclude that it has a unique solution $p_\epsilon \in V$. We shall see that $p_\epsilon(v)$ is an adjoint state for $y_\epsilon(v)$. To this respect, it is sufficient to prove the following proposition.

PROPOSITION 3.1. *Let $\epsilon \in]0,1]$. Then for each $v, w \in L_0^2(\Gamma)$, we have*

$$\int_\Gamma w\left[p_\epsilon(v) + \frac{\partial}{\partial \nu} y_\epsilon(v) - h_2\right] d\gamma = \frac{1}{2} J'_\epsilon(v)(w), \tag{8}$$

where $J'_\epsilon(v)$ is the derivative of the cost functional J_ϵ at v, and $J'_\epsilon(v)(w)$ is its value on the vector w.

PROOF. It is easy to see that the derivative mapping $J'_\epsilon(v)$ of the cost functional J_ϵ at v is given for every $q \in L_0^2(\Gamma)$ by

$$J'_\epsilon(v)(q) = 2\int_\Gamma y_\epsilon(q)\,[y_\epsilon(v) - h_1]\, d\gamma + 2\int_\Gamma \frac{\partial}{\partial \nu} y_\epsilon(q)\left[\frac{\partial}{\partial \nu} y_\epsilon(v) - h_2\right] d\gamma. \tag{9}$$

We have the following equalities:

$$\int_\Gamma \frac{\partial}{\partial \nu} y_\epsilon(w) p_\epsilon(v)\, d\gamma = \int_\Gamma [w - \epsilon y_\epsilon(w)]\, p_\epsilon(v)\, d\gamma = \int_\Gamma w p_\epsilon(v)\, d\gamma - \epsilon\int_\Gamma y_\epsilon(w) p_\epsilon(v)\, d\gamma. \tag{10}$$

Now, by using Green formula, we get

$$\begin{aligned} \int_\Gamma \frac{\partial}{\partial \nu} y_\epsilon(w) p_\epsilon(v)\, d\gamma &= \int_\Gamma y_\epsilon(w) \frac{\partial}{\partial \nu} p_\epsilon(v)\, d\gamma \\ &= -\epsilon\int_\Gamma y_\epsilon(w) p_\epsilon(v)\, d\gamma + \int_\Gamma y_\epsilon(w)\,[y_\epsilon(v) - h_1]\, d\gamma \\ &\quad - \epsilon\int_\Gamma y_\epsilon(w)\left[\frac{\partial}{\partial \nu} y_\epsilon(v) - h_2\right] d\gamma. \end{aligned} \tag{11}$$

From the relations (10) and (11), we obtain

$$\int_\Gamma w p_\epsilon(v)\, d\gamma = \int_\Gamma y_\epsilon(w)\,[y_\epsilon(v) - h_1]\, d\gamma - \epsilon \int_\Gamma y_\epsilon(w) \left[\frac{\partial}{\partial\nu}\, y_\epsilon(v) - h_2\right] d\gamma. \tag{12}$$

But at Γ, one has the relation $\epsilon y_\epsilon(w) = w - \frac{\partial}{\partial\nu}\, y_\epsilon(w)$. As a consequence, we get

$$\begin{aligned}\int_\Gamma w p_\epsilon(v)\, d\gamma = \int_\Gamma y_\epsilon(w)\,[y_\epsilon(v) - h_1]\, d\gamma + \int_\Gamma \frac{\partial}{\partial\nu}\, y_\epsilon(w) \left[\frac{\partial}{\partial\nu}\, y_\epsilon(v) - h_2\right] d\gamma \\ - \int_\Gamma w \left[\frac{\partial}{\partial\nu}\, y_\epsilon(v) - h_2\right] d\gamma.\end{aligned} \tag{13}$$

The equality (13) yields to the wanted formula. ∎

3.2. System of Optimality Conditions

As a consequence, we derive the following characterization of the optimal control u_ϵ:

$$\int_\Gamma v \left[p_\epsilon(u_\epsilon) + \frac{\partial}{\partial\nu}\, y_\epsilon(u_\epsilon) - h_2\right] d\gamma \geq 0, \qquad \forall v \in L_0^2(\Gamma). \tag{14}$$

This condition is equivalent to say that there must exist a constant a_ϵ such that $p_\epsilon(u_\epsilon) + \frac{\partial}{\partial\nu}\, y_\epsilon(u_\epsilon) - h_2 = a_\epsilon$, at Γ. This constant may be calculated by integrating the two sides of the previous inequality, and we have

$$a_\epsilon = -\frac{1}{|\Gamma|} \int_\Gamma h_2\, d\gamma. \tag{15}$$

Now, we are ready to state our system of optimality conditions in the closed form given by

$$\begin{aligned} -\Delta p_\epsilon &= 0, \qquad \text{on } \Omega, \\ -\Delta y_\epsilon &= 0, \qquad \text{on } \Omega, \\ \frac{\partial}{\partial\nu}\, p_\epsilon + \epsilon p_\epsilon &= y_\epsilon - h_1 - \epsilon \left(\frac{\partial}{\partial\nu}\, y_\epsilon - h_2\right), \qquad \text{at } \Gamma, \\ \frac{\partial}{\partial\nu}\, y_\epsilon &= -p_\epsilon + h_2 - \frac{1}{|\Gamma|}\int_\Gamma h_2\, d\gamma, \qquad \text{at } \Gamma, \\ p_\epsilon \in H^1(\Omega), \quad y_\epsilon \in H^1(\Omega), &\qquad \int_\Gamma p_\epsilon\, d\gamma = 0, \quad \text{and} \quad \int_\Gamma y_\epsilon\, d\gamma = 0. \end{aligned} \tag{OP_ϵ}$$

4. CONVERGENCE OF THE ADJOINT STATE p_ϵ

The purpose of this section is to prove the following theorem.

THEOREM 4.1. *Let $\mathcal{U}_{ad} = L_0^2(\Gamma)$, and let $0 < \epsilon < 1$. Let u be the optimal control described by (1.3.2) of Theorem 1.3. Then we have the following.*

(4.1.1) *The net $(p_\epsilon)_\epsilon$ of adjoint states converges strongly in V, to the unique element $p(u) \in V$, verifying*

$$\begin{aligned} -\Delta p(u) &= 0, \qquad \text{on } \Omega, \\ \frac{\partial}{\partial\nu}\, p(u) &= y(u) - h_1 + \frac{1}{|\Gamma|}\int_\Gamma h_1\, d\gamma, \\ p(u) \in H^1(\Omega), &\qquad \int_\Gamma p(u)\, d\gamma = 0. \end{aligned} \tag{$(P_0)^*(u)$}$$

Thus, $p(u)$ is the adjoint state corresponding to the state $y(u)$ solution of the problem $(P_0)(u)$.

(4.1.2) *The optimal control u is characterized by the following system of optimalities:*

$$\begin{aligned} -\Delta p(u) &= 0, \qquad \text{on } \Omega, \\ -\Delta y(u) &= 0, \qquad \text{on } \Omega, \\ \frac{\partial}{\partial \nu} p(u) &= y - h_1, \qquad \text{at } \Gamma, \\ \frac{\partial}{\partial \nu} y(u) &= -p(u) + h_2 - \frac{1}{|\Gamma|}\int_\Gamma h_2 \, d\gamma \qquad \text{at } \Gamma, \\ p(u) \in H^1(\Omega), \quad y(u) \in H^1(\Omega), &\qquad \int_\Gamma p(u)\, d\gamma = 0, \quad \text{and} \quad \int_\Gamma y(u)\, d\gamma = 0. \end{aligned} \tag{OP_0}$$

PROOF. We know (see [3], for example) that we can find a positive constant $\rho > 0$, such that

$$\left[\int_\Omega |\nabla y|^2 \, d\omega + \int_\Gamma y^2 \, d\gamma\right]^{1/2} \le \rho \left[\int_\Omega |\nabla y|^2 \, d\omega\right]^{1/2}, \qquad \forall y \in V. \tag{16}$$

Now, by using the variational formulations for the adjoint systems $(P_\epsilon)^*(u_\epsilon)$ and $(P_0)^*(u)$, we get after some computations the following inequality:

$$\int_\Omega |\nabla(p_\epsilon - p)|^2 \, d\omega = -\epsilon \int_\Gamma p_\epsilon [p_\epsilon - p] \, d\gamma + \int_\Gamma [y_\epsilon - y][p_\epsilon - p] \, d\gamma - \epsilon \int_\Gamma \left[\frac{\partial}{\partial \nu} y_\epsilon - h_2\right] [p_\epsilon - p] \, d\gamma, \tag{17}$$

where we have denoted $p_\epsilon := p_\epsilon(u_\epsilon)$ and $p := p(u)$. With the help of the relation (16), we obtain

$$\begin{aligned} \int_\Omega |\nabla(p_\epsilon - p)|^2 \, d\omega \le \epsilon\rho \|p_\epsilon\|_{L^2(\Gamma)} \|p_\epsilon - p\|_V + \rho^2 \|y_\epsilon - y\|_V \|p_\epsilon - p\|_V \\ + \epsilon\rho \|p_\epsilon - p\|_V \|u_\epsilon - \epsilon y_\epsilon - h_2\|_{L^2(\Gamma)}. \end{aligned} \tag{18}$$

The inequality (18) is equivalent to say that we have

$$\|p_\epsilon - p\|_V \le \epsilon\rho \, \|p_\epsilon\|_{L^2(\Gamma)} + \rho^2 \, \|y_\epsilon - y\|_V + \epsilon\rho \, \|u_\epsilon - \epsilon y_\epsilon - h_2\|_{L^2(\Gamma)}. \tag{19}$$

The last inequality finishes the proof of our theorem. ∎

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